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An extension of workload capacity space for systems with more than two channels[☆]



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HIGHLIGHTS

- Extends the capacity coefficient bounds for AND and OR tasks to $n \geq 2$ channels.
- Introduces capacity coefficient bounds for single-target self-terminating tasks.
- Defines capacity coefficient bounds for identically distributed channels.

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ABSTRACT

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1. Introduction

The study of the combination of multiple sources of information is ubiquitous in cognitive psychology. Examples include visual and memory search tasks, in which the multiple sources are the array items through which a participant must search, and complex decision making tasks in which multiple types of information must be combined to make a good decision. One question that often arises is the extent to which adding more sources of information affects the processing of each individual source. For example, one might inquire whether it takes longer to determine the presence of a particular object in a stimulus when there are more total objects in the stimulus. In this paper, we refer to a cognitive system's response to variations in the number of information sources as its workload capacity.

One of the most commonly used measures of workload capacity is the Race Model Inequality (Miller, 1982), which gives an upper bound on the response speed of a parallel processing model with context invariance (defined below) for testing one versus two sources of information using cumulative distribution functions (CDFs) in the context of minimum-time, redundant target decisions. Subsequent to Miller's paper, the basic logic of the Race Model Inequality has been extended to a lower bound on minimum-time models as well as upper and lower CDF bounds for other stopping criteria (e.g., all information must be processed rather than any one source) and more sources of information (Colonius & Vorberg, 1994; Grice, Canham, & Gwynne, 1984). Using a stronger set of assumptions, together with a well-defined baseline model, Townsend and colleagues derived an equality to test workload capacity, termed the capacity coefficient (Blaha & Townsend, submitted for publication; Townsend & Nozawa, 1995; Townsend & Wenger, 2004).

Recently, Townsend and Eidels (2011) introduced the notion of a unified workload capacity space for plotting both the capacity coefficient and the CDF bounds on standard parallel processing on the same plot space. This work served to transform the upper and lower bounds on parallel processing from probability space

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(ordinate values bounded on $[0, 1]$) into the same unit-less axis as the capacity coefficient, with ordinate values bounded on $[0, +\infty)$. Practically, this unified space allows investigators to directly compare in the same plot capacity coefficient values with the bounds on standard parallel processing, which enables some estimation of possible extreme capacity values (very high super capacity, very low limited capacity), as well as some inferences about possible model architectures (e.g. violation of the race model with super capacity implies a possible coactive model architecture). Unfortunately, Townsend and Eidels (2011) limited their derivations to models with only two possible operating channels. The capacity coefficients are defined for $n \geq 2$ channels (Townsend & Wenger, 2004), as are the CDF bounds on standard parallel processing (Colonius & Vorberg, 1994), so the restriction to $n = 2$ channels is an unnecessary limitation of the applicability for the new unified space.

Herein, we complete the derivation of the unified workload capacity space by extending the transformations of the parallel model bounds to the general case of n channels, where $n \geq 2$. We also provide the alternative versions for the unified space when the marginal distributions of the channels are assumed to be independent and identically distributed (IID), which serves to simplify the computations. Finally, in addition to the AND and OR cases derived in previous work, we add the bounds for single-target self-terminating processing, recently introduced in Blaha (2010) and Blaha and Townsend (submitted for publication).

We use the following notation throughout the paper. Let $F_c(t) = P[T_c \leq t]$ be the CDF of response times for a system with the set of n active channels, $\mathcal{C} = \{1, \dots, n\}$. To denote the CDF of a single channel c among the \mathcal{C} channels, we use $F_{c,\mathcal{C}}(t)$, and to denote the processing of a single channel c alone (i.e. no other active channels in the model or $n = 1$), we use $F_c(t)$. We use set minus notation $\mathcal{C} \setminus \{c\}$ to indicate the full set of channels \mathcal{C} except c .¹

In this work, standard parallel processing is used to refer to a processing system that exhibits independent channel distributions (no cross-talk, no statistical facilitation/degradation). This means that for any number of active channels, the CDF for all channels active simultaneously is the product of the marginal distributions, $F_{\mathcal{C}}(t) = \prod_{c=1}^n F_{c,\mathcal{C}}(t)$. Additionally, standard parallel processing exhibits context independence, or context invariance. This means that the marginal distribution of any given channel c is identically distributed when any number of additional channels are also operating. We denote this by $F_c(t) = F_{c,\mathcal{C}}(t)$. Functionally, this allows the individual channels to be estimated by single-target or single-feature conditions in an experiment, which often greatly simplifies the number of conditions the experimenter needs to test in order to use these models.

Additionally, we note that standard parallel processing is often referred to as the parallel race model, the parallel horse-race model, or simply the race model (see, for example, Miller, 1982). This analogy specifically refers to the case when the first channel to finish processing is enough to make a response, which results in a minimum time response. This is the case of minimum time processing, also termed first-terminating stopping or an OR (logical OR-gate) stopping rule. This would be the stopping rule engaged in tasks like visual search among redundant targets (no distractors) where the identification of the first target to be searched is enough to complete the task.

A standard parallel model architecture can be defined under other stopping rules as well. These include exhaustive stopping (last-terminating or logical AND stopping) and the single-target self-terminating (ST-ST) stopping. Under an exhaustive stopping rule, all channels must complete processing before a response is made, resulting in maximum time (rather than minimum time) responses (i.e. the response time is determined by the slowest channel). Under an ST-ST stopping rule, the completion of a specific, single target channel is enough to terminate the processing, but the target channel may be any of the n possible channels to be processed—first, last, or somewhere in between.

We can ground our intuition for these various stopping rules by considering again a visual search task. As described above, the OR or minimum-time stopping rule applies in cases of redundant targets with no distractors, such that a decision can be made on the first item to complete processing in the search array. If the search array contains no targets and only distractor items, then the observer must use an exhaustive stopping rule to completely search all distractors in order to rule out the presence of a target. The response time is determined by the slowest item in the array to be processed. In the conditions where the search array is comprised of a mix of distractors and targets, then the ST-ST stopping rule can be invoked. This is because the observer can make a response as soon as any single target is identified. On some trials, this may be the first item to process; on others, it may be the last, and sometimes it will be an item processed in between the first and last.

Importantly, each of these stopping rules changes the form of the capacity coefficient and the predictions of the race model bounds, so we will present the derivation of the bounds in unified workload capacity space for each model in turn. Experimenters, having ground-truth knowledge of the experimental conditions and the stopping rules they should invoke,² can apply the appropriate capacity coefficients to the subsets of data for each stopping rule condition.

Before we get into the derivation of the unified response time bounds, we want to remind readers that all CDFs and survivor functions exist in the range $[0, 1]$, so the natural logarithm of those functions produce negative values. Thus, cumulative reverse hazard functions (natural log of the CDF) exist on the range $(-\infty, 0]$, as do the natural logarithms of any bounds formed by a single CDF or products of CDFs (sums of CDFs can range above 1, and so the natural log can exist on $(-\infty, +\infty)$). These negative values will influence the derivation of inequality chains throughout this paper. The cumulative hazard function used in the minimum time bounds, is the negative natural log of the survivor function, and its range is $[0, +\infty)$, leaving fewer negative signs to track in those proofs.

2. Minimum time bounds

Let $F_{\mathcal{C}}(t) = P[\min_{\mathcal{C}}(T_c) \leq t]$, for all real $t \geq 0$ and $c \in \mathcal{C}$, be the CDF for an n -channel system operating under a minimum time stopping rule, where $\mathcal{C} = \{1, \dots, n\}$ is the set of all possible channels. Define $F_{\mathcal{C} \setminus \{i\}}(t) = P[\min_{\mathcal{C} \setminus \{i\}} T_c \leq t]$ as the CDF if all channels except i are running, and define $F_{\mathcal{C} \setminus \{i,j\}}(t) = P[\min_{\mathcal{C} \setminus \{i,j\}} T_c \leq t]$, $i \neq j$, for the CDF of all channels but i and j . Further, we denote the survivor function with $S_{\mathcal{C}}(t) = 1 - F_{\mathcal{C}}(t)$.

¹ We acknowledge that set notation has not typically been used in this context previously; rather, various CDFs, hazard functions, etc., have employed combinations of superscripts and subscripts to note the total number of channels in the set and number of channels actively processing. In our experience, when both superscripts and subscripts are employed and then additional exponents are needed in multiplication, the previous notation can get unwieldy.

² Generally, the design of the experiment will dictate the stopping rule that people can engage to sufficiently complete the task. However, people may not engage the expected stopping rule (e.g. Johnson, Blaha, Houtp, & Townsend, 2010). Additional aspects of Systems Factorial Technology modeling can be utilized to assess the stopping rules actually engaged in a task, but explication of those tools is beyond our present scope (cf. Houtp, Blaha, McIntire, Havig, & Townsend, 2014; Townsend & Nozawa, 1995).

We measure the amount of work completed in each channel c with the cumulative hazard function, defined as:

$$H_c(t) = \int_0^t \frac{f_c(\tau)}{S_c(\tau)} d\tau = -\ln\{S_c(t)\}$$

which can easily be estimated directly from the empirical response time survivor function for any experimental condition (see also Luce, 1986).

The capacity coefficient for minimum time (first-terminating, OR) processing for an n -channel model is defined as a ratio of cumulative hazard functions (Townsend & Nozawa, 1995; Townsend & Wenger, 2004):

$$C_{OR}(t) = \frac{H_c(t)}{\sum_{c=1}^n H_c(t)}. \quad (1)$$

The numerator in Eq. (1) is the observed minimum-time processing of all n active channels, $H_c(t) = -\ln\{S_c(t)\}$, while the denominator is the prediction of a benchmark standard parallel processing model, exhibiting independence and, in this terminology, unlimited capacity, $\sum_{c=1}^n H_c(t) = -\ln\{S_1(t)S_2(t) \cdots S_n(t)\}$. Thus, if performance were equal to that predicted by the baseline model the ratio would be equal to one. In this case, the performance is referred to as unlimited capacity. $C_{OR}(t) > 1$ indicates super capacity, or better-than-benchmark performance. $C_{OR}(t) < 1$ indicates limited capacity, i.e., worse-than-benchmark performance.³

The original race model CDF bound by Miller (1982) provided an upper bound on the CDF from the parallel, minimum-time model with $n = 2$ channels given by $F_{[A,B]}(t) \leq F_A(t) + F_B(t)$, where A, B denote the two parallel channels. This inequality arises from the inclusion-exclusion principle. In particular, if the response time is determined by the faster of the two racing processes, then a response has occurred by a given time only if at least one of processes has finished. Hence, under the assumption of context invariance,

$$P[T_A \leq t \text{ or } T_B \leq t] = P[T_A \leq t] + P[T_B \leq t] - P[T_A \leq t \text{ and } T_B \leq t].$$

The last term is not directly measurable in standard OR experiments, but because the term must be positive we can drop it and change the equality to an inequality.

Grice et al. (1984) introduced the concept of a lower bound for the same processing model, which is defined as $F_{[A,B]}(t) \geq \max[F_A(t), F_B(t)]$. The logic of the Grice bound is that in a race, the probability that the faster of the two channels has finished is at least as high as the probability that either of the individual channels has finished:

$$P[T_A \leq t \text{ or } T_B \leq t] \geq P[T_A \leq t] \quad \text{and} \\ P[T_A \leq t \text{ or } T_B \leq t] \geq P[T_B \leq t].$$

Colonius and Vorberg (1994) provided the n -channel generalization of both CDF bounds on parallel minimum-time processing in the inequality chain

$$\max_i [F_{c \setminus \{i\}}(t)] \leq F_c(t) \\ \leq \min_{i,j} [F_{c \setminus \{i\}}(t) + F_{c \setminus \{j\}}(t) - F_{c \setminus \{i,j\}}(t)]. \quad (2)$$

Theorem 1. The unified workload capacity space inequality chain for the capacity of an n -channel, minimum-time system is, for $i \neq j$,

$$\frac{\ln \left\{ \min_i [S_{c \setminus \{i\}}(t)] \right\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}} \\ \leq C_{OR}(t) \leq \frac{\ln \left\{ \max_{i,j} [S_{c \setminus \{i\}}(t) + S_{c \setminus \{j\}}(t) - S_{c \setminus \{i,j\}}(t)] \right\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}}. \quad (3)$$

Proof. From Eq. (1), $C_{OR}(t) \ln \left\{ \prod_{c=1}^n S_c(t) \right\} = \ln\{S_c(t)\}$. Rewrite the upper bound from Eq. (2) in terms of the survivor functions to get

$$S_c(t) \geq \max_{i,j} [S_{c \setminus \{i\}}(t) + S_{c \setminus \{j\}}(t) - S_{c \setminus \{i,j\}}(t)].$$

It follows that

$$C_{OR}(t) \ln \left\{ \prod_{c=1}^n S_c(t) \right\} \\ \geq \ln \left\{ \max_{i,j} [S_{c \setminus \{i\}}(t) + S_{c \setminus \{j\}}(t) - S_{c \setminus \{i,j\}}(t)] \right\} \\ \Rightarrow C_{OR}(t) \leq \frac{\ln \left\{ \max_{i,j} [S_{c \setminus \{i\}}(t) + S_{c \setminus \{j\}}(t) - S_{c \setminus \{i,j\}}(t)] \right\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}}.$$

Similarly, rewrite the lower bound of Eq. (2) as

$$\min_i [S_{c \setminus \{i\}}(t)] \geq S_c(t)$$

and it follows that

$$C_{OR}(t) \ln \left\{ \prod_{c=1}^n S_c(t) \right\} \leq \ln \left\{ \min_i [S_{c \setminus \{i\}}(t)] \right\} \\ \Rightarrow C_{OR}(t) \geq \frac{\ln \left\{ \min_i [S_{c \setminus \{i\}}(t)] \right\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}}. \quad \square$$

Under the assumption that the marginal distributions for each channel are IID, then all $F_{c \setminus \{i\}}(t)$ are the same for any choice of $i \in \mathcal{C}$ and we can write this as $F_{c-1}(t)$ to denote the CDF for $n - 1$ active channels. Similarly, the IID assumption means the $F_{c \setminus \{i,j\}}(t)$ are the same for any choice of $i, j \in \mathcal{C}$, and we write this as $F_{c-2}(t)$ for the CDF with $n - 2$ active channels. Consequently, Eq. (2) simplifies to (Colonius & Vorberg, 1994)

$$F_{c-1}(t) \leq F_c(t) \leq [2F_{c-1}(t) - F_{c-2}(t)]. \quad (4)$$

Corollary 1. When the marginal distributions of the parallel model are IID, the unified workload capacity space inequality chain for the capacity of an n -channel, minimum-time system is defined by

$$\frac{\ln\{S_{c-1}(t)\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}} \leq C_{OR}(t) \leq \frac{\ln\{2S_{c-1}(t) - S_{c-2}(t)\}}{\ln \left\{ \prod_{c=1}^n S_c(t) \right\}}. \quad (5)$$

The proof of Corollary 1 is similar to the proof of Theorem 1.

³ Statistical tests for $C_{OR}(t)$ are available (Houtp & Townsend, 2012), but the details are beyond the scope of this paper.

3. Maximum time bounds

Let $G_c(t) = P[\max_{\mathcal{C}}(T_c) \leq t]$, where again $\mathcal{C} = \{1, \dots, n\}$ is the set of all n channels and $c \in \mathcal{C}$, be the cumulative distribution function of response times for an n -channel system under a maximum time (logical AND, exhaustive) stopping rule.⁴

In order for the capacity coefficient inferences to be consistent with those for Eq. (1), we utilize the cumulative reverse hazard function to measure the work throughput for each channel under the maximum-time stopping rule (for a full discussion of the reasoning, see Townsend & Eidels, 2011, Townsend & Wenger, 2004). The cumulative reverse hazard function for processing channel c is given by

$$K_c(t) = \int_0^t \frac{g_c(\tau)}{G_c(\tau)} d\tau = \ln\{G_c(t)\}$$

which, again, can easily be estimated directly from the empirical response time CDF for any experimental condition.

The capacity coefficient for maximum time processing is defined as (Townsend & Wenger, 2004)

$$C_{\text{AND}}(t) = \frac{\sum_{c=1}^n K_c(t)}{K_c(t)}. \quad (6)$$

The numerator in the AND case is the prediction of the benchmark unlimited capacity, independent parallel model, $\sum_{c=1}^n K_c(t) = \ln\{F_1(t)F_2(t) \cdots F_n(t)\}$, while the denominator is the observed processing of all n channels under the maximum-time stopping rule, $K_c(t) = \ln\{F_c(t)\}$. Capacity inferences, again, are relative to the value $C_{\text{AND}}(t) = 1$, which indicates unlimited capacity. $C_{\text{AND}}(t) > 1$ indicates super capacity processing, and $C_{\text{AND}}(t) < 1$ indicates limited capacity processing.

We can motivate bounds for AND processing by following logic similar to the Miller and Grice bounds, illustrated here on two channels, A and B . If the AND process has not yet finished, then at least one of the two channels must not yet be finished,

$$\begin{aligned} 1 - P[T_A \leq t \text{ and } T_B \leq t] \\ &= P[T_A > t \text{ or } T_B > t] \\ &= P[T_A > t] + P[T_B > t] - P[T_A > t \text{ and } T_B > t]. \end{aligned}$$

Dropping the final term, we get

$$\begin{aligned} 1 - P[T_A \leq t \text{ and } T_B \leq t] &\leq P[T_A > t] + P[T_B > t] \\ &= 1 - P[T_A \leq t] + 1 - P[T_B \leq t]. \end{aligned}$$

Rearranging the terms gives the lower bound for a two channel AND process.

For the upper bound, note that in an AND task the response time is determined by the slower of the two processes, so if a response was made by a given time, then *both* processes must have completed.

The general bounds for n exhaustively processed channels are (Colonius & Vorberg, 1994):

$$\begin{aligned} \max_{i,j} [G_{\mathcal{C} \setminus \{i\}}(t) + G_{\mathcal{C} \setminus \{j\}}(t) - G_{\mathcal{C} \setminus \{i,j\}}(t)] \\ \leq G_c(t) \leq \min_i [G_{\mathcal{C} \setminus \{i\}}(t)]. \end{aligned} \quad (7)$$

Theorem 2. *The unified workload capacity space inequality chain for the capacity of an n -channel, maximum-time system is, for $i \neq j$,*

$$\begin{aligned} \frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln \left\{ \max_{i,j} [G_{\mathcal{C} \setminus \{i\}}(t) + G_{\mathcal{C} \setminus \{j\}}(t) - G_{\mathcal{C} \setminus \{i,j\}}(t)] \right\}} \\ \leq C_{\text{AND}}(t) \leq \frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln \left\{ \min_i [G_{\mathcal{C} \setminus \{i\}}(t)] \right\}}. \end{aligned} \quad (8)$$

Proof. From Eq. (6), $C_{\text{AND}}(t) \ln\{G_c(t)\} = \ln\{\prod_{c=1}^n G_c(t)\}$. Utilizing Eq. (7), it follows that, for the upper bound

$$\begin{aligned} C_{\text{AND}}(t) \ln\{G_c(t)\} &\leq C_{\text{AND}}(t) \ln \left\{ \min_i [G_{\mathcal{C} \setminus \{i\}}(t)] \right\} \\ \Rightarrow \ln \left\{ \prod_{c=1}^n G_c(t) \right\} &\leq C_{\text{AND}}(t) \ln \left\{ \min_i [G_{\mathcal{C} \setminus \{i\}}(t)] \right\} \\ \Rightarrow \frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln \left\{ \min_i [G_{\mathcal{C} \setminus \{i\}}(t)] \right\}} &\geq C_{\text{AND}}(t). \end{aligned}$$

Similarly, for the lower bound,

$$\begin{aligned} C_{\text{AND}}(t) \ln\{G_c(t)\} \\ &\geq C_{\text{AND}}(t) \ln \left\{ \max_{i,j} [G_{\mathcal{C} \setminus \{i\}}(t) + G_{\mathcal{C} \setminus \{j\}}(t) - G_{\mathcal{C} \setminus \{i,j\}}(t)] \right\} \\ \Rightarrow \ln \left\{ \prod_{c=1}^n G_c(t) \right\} \\ &\geq C_{\text{AND}}(t) \ln \left\{ \max_{i,j} [G_{\mathcal{C} \setminus \{i\}}(t) + G_{\mathcal{C} \setminus \{j\}}(t) - G_{\mathcal{C} \setminus \{i,j\}}(t)] \right\} \\ \Rightarrow \frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln \left\{ \max_{i,j} [G_{\mathcal{C} \setminus \{i\}}(t) + G_{\mathcal{C} \setminus \{j\}}(t) - G_{\mathcal{C} \setminus \{i,j\}}(t)] \right\}} &\leq C_{\text{AND}}(t). \quad \square \end{aligned}$$

Under the assumption that the marginal distributions for each channel are IID, for any choice of $i \in \mathcal{C}$, all $G_{\mathcal{C} \setminus \{i\}}(t)$ are the same and for any choice of $i, j \in \mathcal{C}$, all $G_{\mathcal{C} \setminus \{i,j\}}(t)$ are the same. We write these as $G_{c-1}(t)$ and $G_{c-2}(t)$, for $n-1$ and $n-2$ active processing channel systems, respectively. It follows that Eq. (7) simplifies to (Colonius & Vorberg, 1994):

$$[2G_{c-1}(t) - G_{c-2}(t)] \leq G_c(t) \leq G_{c-1}(t). \quad (9)$$

Corollary 2. *When the marginal distributions of the parallel model are IID, the unified workload capacity space inequality chain for the capacity of an n -channel, maximum-time system is defined by*

$$\frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln\{2G_{c-1}(t) - G_{c-2}(t)\}} \leq C_{\text{AND}}(t) \leq \frac{\ln \left\{ \prod_{c=1}^n G_c(t) \right\}}{\ln\{G_{c-1}(t)\}}. \quad (10)$$

The proof of Corollary 2 is similar to the proof of Theorem 2.

4. Single-target self-terminating bounds

Blaha (2010) recently introduced a new capacity coefficient for ST-ST processing, with full details explicated in Blaha and Townsend (submitted for publication). For completeness with

⁴ Note that the change in notation here is to simply help the reader to distinguish the CDFs for minimum- and maximum-time stopping rules.

Table 1
Summary of all bounds on the capacity coefficient.

Stopping rule	n channels	n IID channels	2 channels
Lower bounds			
OR	$\frac{\ln\{\min_i[S_{e\setminus\{i\}}(t)]\}}{\ln\{\prod_{c=1}^n S_c(t)\}}$	$\frac{\ln[S_{e-1}(t)]}{\ln\{\prod_{c=1}^n S_c(t)\}}$	$\frac{\ln[\min\{S_1(t), S_2(t)\}]}{\ln[S_1(t)S_2(t)]}$
ST-ST	$\frac{\ln[F_k(t)]}{\sum_{c=1}^n \ln[F_c(t)]}$	$\frac{\ln[F_k(t)]}{n \ln[F_c(t)]}$	$\frac{\ln[F_k(t)]}{\ln[F_1(t)F_2(t)]}$
AND	$\frac{\ln\{\prod_{c=1}^n G_c(t)\}}{\ln\{\max_{i,j}[G_{e\setminus\{i\}}(t)+G_{e\setminus\{j\}}(t)-G_{e\setminus\{i,j\}}(t)]\}}$	$\frac{\ln\{\prod_{c=1}^n G_c(t)\}}{\ln[2G_{e-1}(t)-G_{e-2}(t)]}$	$\frac{\ln[G_1(t)G_2(t)]}{\ln[G_1(t)+G_2(t)-1]}$
Upper bounds			
OR	$\frac{\ln\{\max_{i,j}[S_{e\setminus\{i\}}(t)+S_{e\setminus\{j\}}(t)-S_{e\setminus\{i,j\}}(t)]\}}{\ln\{\prod_{c=1}^n S_c(t)\}}$	$\frac{\ln[2S_{e-1}(t)-S_{e-2}(t)]}{\ln\{\prod_{c=1}^n S_c(t)\}}$	$\frac{\ln[S_1(t)+S_2(t)-1]}{\ln[S_1(t)S_2(t)]}$
ST-ST	$\frac{\ln[F_k(t)]}{\ln\{\sum_{c=1}^n F_c(t)\}}$	$\frac{\ln[F_k(t)]}{\ln[nF_c(t)]}$	$\frac{\ln[F_k(t)]}{\ln[F_1(t)+F_2(t)]}$
AND	$\frac{\ln\{\prod_{c=1}^n G_c(t)\}}{\ln\{\min_i[G_{e\setminus\{i\}}(t)]\}}$	$\frac{\ln\{\prod_{c=1}^n G_c(t)\}}{\ln[G_{e-1}(t)]}$	$\frac{\ln[G_1(t)G_2(t)]}{\ln[\min\{G_1(t), G_2(t)\}]}$

respect to the results in Townsend and Eidels (2011), we here give the ST-ST parallel processing CDF bounds for both $n = 2$ -channel models and $n \geq 2$ -channel models.

Let $F_{k,e}(t) = P[T_{k,e} \leq t]$ denote the CDF of response times for target channel $k \in \mathcal{C}$. Let $K_{k,e}(t) = \int_{\tau=0}^t \frac{f_{k,e}(\tau)}{F_{k,e}(\tau)} d\tau = \ln\{F_{k,e}(t)\}$ be the cumulative reverse hazard function for target channel $k \in \mathcal{C}$.

The capacity coefficient for ST-ST processing is defined as (Blaha, 2010; Blaha & Townsend, submitted for publication)

$$C_{\text{STST}}(t) = \frac{K_k(t)}{K_{k,e}(t)}. \quad (11)$$

The benchmark parallel model is in the numerator of Eq. (11), $K_k(t) = \ln\{F_k(t)\}$, and the observed processing of target channel k among n active channels is in the denominator, $K_{k,e}(t) = \ln\{F_{k,e}(t)\}$. The inferences about unlimited, limited, and super capacity are the same as the OR and AND models. We note that the baseline UCIP model for ST-ST processing exploits the fact that under the stochastic independence and unlimited capacity assumptions, the distribution of response times for processing single target channel k should be the same regardless of the number of other items to be processed. Consequently, the UCIP model prediction is derived from the processing of target k alone.

The bounds on ST-ST processing are

$$\prod_{c=1}^n F_c(t) \leq F_{k,e}(t) \leq \sum_{c=1}^n F_c(t). \quad (12)$$

For $n = 2$ channels, with the two channels denoted $\mathcal{C} = \{1, 2\}$, the bounds simplify to

$$F_1(t)F_2(t) \leq F_{k,e}(t) \leq F_1(t) + F_2(t).$$

Theorem 3. The unified workload capacity space inequality chain for the capacity of an n -channel, single-target self-terminating system is,

$$\frac{\ln\{F_k(t)\}}{\sum_{c=1}^n \ln\{F_c(t)\}} \leq C_{\text{STST}}(t) \leq \frac{\ln\{F_k(t)\}}{\ln\left\{\sum_{c=1}^n F_c(t)\right\}}. \quad (13)$$

The proof of Theorem 3 is nearly identical to the proof of Theorem 2, substituting the capacity coefficient and bounds for ST-ST capacity in for those of maximum-time processing.

Under the assumption that the marginal distributions for each channel are IID, we use the CDF of a single channel $c \in \mathcal{C}$, and rewrite Eq. (12) as

$$[F_c(t)]^n \leq F_{k,e}(t) \leq nF_c(t). \quad (14)$$

Corollary 3. When the marginal distributions are IID, the unified capacity space bounds for ST-ST processing are

$$\frac{\ln\{F_k(t)\}}{n \ln\{F_c(t)\}} \leq C_{\text{STST}}(t) \leq \frac{\ln\{F_k(t)\}}{\ln\{nF_c(t)\}}. \quad (15)$$

The proof of this is straightforward so it is not included.

5. Conclusion

We have provided the straight forward extension of the unified workload capacity space bounds for standard parallel processing from the limited existing definitions for $n = 2$ channels given in Townsend and Eidels (2011) to the full $n \geq 2$ -channel situation for minimum-time, maximum-time, and single-target self-terminating stopping rules. The full set of bounds, including all special cases considered to date, are summarized in Table 1. This extension enables powerful generalizations of this approach to multiple stopping rules and any number of channels of interest, in order to model the complete processing mechanisms for an experiment of interest. Mapping the bounds onto the unified capacity space for any number of channels enables a single plot to be used to compare the capacity coefficient values to the upper and lower bounds on standard parallel processing in order to make more direct inferences about extreme capacity values.

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